

A Topological Approach to Chemical Organizations - SUPPLEMENT

Gil Benkö^a, Florian Centler^b, Peter Dittrich^b,
Christoph Flamm^{a,d}, Bärbel M. R. Stadler^c, and
Peter F. Stadler^{a,d,e}

^a*Bioinformatics Group, Department of Computer Science, and
Interdisciplinary Center for Bioinformatics, University of Leipzig,
Härtelstraße 16-18, D-04107 Leipzig, Germany
{gil,studla,xtof}@bioinf.uni-leipzig.de*

^b*Bio Systems Analysis Group, Jena Center for Bioinformatics and Department of
Mathematics and Computer Science, Friedrich Schiller University Jena, D-07743
Jena, Germany
{dittrich}@minet.uni-jena.de*

^c*Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103, Leipzig, Germany
{stadler}@mis.mpg.de*

^d*Department of Theoretical Chemistry, University of Vienna,
Währingerstraße 17, A-1090 Wien, Austria
{studla,xtof}@tbi.univie.ac.at*

^e*The Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe NM 87501*

Abstract

This supplement extends Section 7 on the main text, providing additional material on generalized topologies and pertinent references.

1 Further Remarks on Generalized Topologies

This text summarizes, and in part slightly extends, results by Day [5], Hammer [7, 11] and Gnifka [8] on structures defined by a set X endowed with an arbitrary set-valued set-function.

Let c' and c'' be two generalized closure operators on X . We say that c' is *finer* than c'' , $c' \succeq c''$, or c'' is *coarser* than c' if $c'(A) \subseteq c''(A)$ for all $A \in \mathcal{P}(X)$. Note that $c' \succeq c''$ and $c' \preceq c''$ implies $c' = c''$.

A function $f : (X, \text{cl}) \rightarrow (Y, \text{cl})$ is

closure preserving if for all $A \in \mathcal{P}(X)$, $f(\text{cl}(A)) \subseteq \text{cl}(f(A))$ holds;

continuous if for all $B \in \mathcal{P}(Y)$, $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ holds.

It is obvious that the identity $\iota : (X, \text{cl}) \rightarrow (X, \text{cl}) : x \mapsto x$ is both closure-preserving and continuous since $\iota(\text{cl}(A)) = \text{cl}(A) \subseteq \text{cl}(A) = \text{cl}(\iota(A))$. Furthermore, the concatenation $h = g \circ f$ of the closure-preserving (continuous) functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is again closure-preserving (continuous).

Let (X, cl) and (Y, cl) be two sets with arbitrary closure functions and let $f : X \rightarrow Y$. Then the following conditions (for continuity) are equivalent, see e.g. [9, Thm.3.1.]:

- (i) $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for all $B \in \mathcal{P}(Y)$.

Table 1

Basic axioms for Generalized Topologies.

The properties below are meant to hold for all $A, B \in \mathcal{P}(X)$ and all $x \in X$, respectively.

	closure	interior	neighborhood
K0'	$\exists A : x \notin \text{cl}(A)$	$\exists A : x \in \text{int}(A)$	$\mathcal{N}(x) \neq \emptyset$
K0	$\text{cl}(\emptyset) = \emptyset$	$\text{int}(X) = X$	$X \in \mathcal{N}(x)$
K1 isotonic, monotone	$A \subseteq B \implies \text{cl}(A) \subseteq \text{cl}(B)$ $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$ $\text{cl}(A) \cup \text{cl}(B) \subseteq \text{cl}(A \cup B)$	$A \subseteq B \implies \text{int}(A) \subseteq \text{int}(B)$ $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$ $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$	$N \in \mathcal{N}(x)$ and $N \subseteq N' \implies N' \in \mathcal{N}(x)$
KA	$\text{cl}(X) = X$	$\text{int}(\emptyset) = \emptyset$	$\emptyset \notin \mathcal{N}(x)$
KB	$A \cup B = X \implies \text{cl}(A) \cup \text{cl}(B) = X$	$A \cap B = \emptyset \implies \text{int}(A) \cap \text{int}(B) = \emptyset$	$N', N'' \in \mathcal{N}(x) \implies N' \cap N'' \neq \emptyset$
K2 expansive	$A \subseteq \text{cl}(A)$	$\text{int}(A) \subseteq A$	$N \in \mathcal{N}(x) \implies x \in N$
K3 sub-linear	$\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$	$\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cup B)$	$N', N'' \in \mathcal{N}(x) \implies N' \cap N'' \in \mathcal{N}(x)$
K4 idempotent	$\text{cl}(\text{cl}(A)) = \text{cl}(A)$	$\text{int}(\text{int}(A)) = \text{int}(A)$	$N \in \mathcal{N}(x) \iff \text{int}(N) \in \mathcal{N}(x)$
K5 additive	$\bigcup_{i \in I} \text{cl}(A_i) = \text{cl}\left(\bigcup_{i \in I} A_i\right)$	$\bigcap_{i \in I} \text{int}(A_i) = \text{int}\left(\bigcap_{i \in I} A_i\right)$	$\mathcal{N}(x) = \emptyset$ or $\exists N(x) : N \in \mathcal{N}(x) \iff N(x) \subseteq N$

(ii) $f^{-1}(\text{int}(B)) \subseteq \text{int}(f^{-1}(B))$ for all $B \in \mathcal{P}(Y)$.(iii) $B \in \mathcal{N}(f(x))$ implies $f^{-1}(B) \in \mathcal{N}(x)$ for all $B \in \mathcal{P}(Y)$ and all $x \in X$.The conditions (iii) can be used as a definition of continuity at each individual point $x \in X$.

The notation of a neighborhood for an individual point can be extended naturally to sets: Let $A \in \mathcal{P}(X)$. A set V is a neighborhood of A , in symbols $V \in \mathcal{N}(A)$, if $V \in \mathcal{N}(x)$ for all $x \in A$. Obviously, $\mathcal{N}(\{x\}) = \mathcal{N}(x)$.

Lemma 1 For all $V, A \in \mathcal{P}(X)$ we have $V \in \mathcal{N}(A)$ if and only if $A \subseteq \text{int}(V)$.

Almost all approaches to extend the framework of topology assume at least that the closure functions are isotonic, or, equivalently, that the neighborhoods of a point form a “stack”, see e.g. [2, 5, 8, 10, 11]. The importance of isotony is emphasized by several equivalent conditions, see e.g. [11, Lem.10]:

(K1) $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$ for all $A, B \in \mathcal{P}(X)$.(K1') $\text{cl}(A) \cup \text{cl}(B) \subseteq \text{cl}(A \cup B)$ for all $A, B \in \mathcal{P}(X)$.(K1'') $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$

A (not necessarily non-empty) collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *stack* if $F \in \mathcal{F}$ and $F \subseteq G$ implies $G \in \mathcal{F}$. Let us write $\mathfrak{S}(X)$ for the set of all stacks. It is important to distinguish the empty set $\emptyset \in \mathcal{P}(X)$ and the empty stack $\emptyset \subseteq \mathcal{P}(X)$. There is a condition equivalent to (K1) in terms of the neighborhood function: The closure function cl is isotonic if and only if $\mathcal{N}(x)$ is a stack for all $x \in X$.

Kuratowski's axioms for the closure function of a topological space [13] may be seen as specializations of the very general closure functions that we have considered so far. Let (X, cl) be a generalized closure space and consider the properties of the closure function for all $A, B \in \mathcal{P}(X)$ that are listed in the first column of table 1. Columns 2 and 3 compile equivalent definitions in terms of interior and neighborhood, respectively.

Different combinations of these axioms define topological structures that have been studied to various degrees in the literature; table 2 summarizes the best known ones.

Table 2

Axioms for various types of closure functions.

Defining axioms are marked by \bullet , further properties that implied are marked by \circ .

Axiom	$\text{cl}(\emptyset) = \emptyset$	$A \subseteq B \implies \text{cl}(A) \subseteq \text{cl}(B)$ isotonic	$A \subseteq \text{cl}(A)$ enlarging	$\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$ sub-linear	$\text{cl}(\text{cl}(A)) = \text{cl}(A)$ idempotent	$\text{cl}(\bigcup_i A_i) = \bigcup_i \text{cl}(A_i)$ additive	Ref.
	(K0)	(K1)	(K2)	(K3)	(K4)	(K5)	
Extended Topology	\bullet	\bullet					[11]
Brissaud	\bullet		\bullet				[2]
Neighborhood space	\bullet	\bullet	\bullet				[12]
Closure space	(\bullet)	\bullet	\bullet		\bullet		[17]
Smyth space	\bullet	\bullet		\bullet			[16]
Binary relation	\bullet	\circ		\circ		\bullet	[3, 14]
Pretopology	\bullet	\bullet	\bullet	\bullet			[4]
Topology	\bullet	\bullet	\bullet	\bullet	\bullet		
Alexandroff space	\bullet	\circ	\bullet	\circ		\bullet	
Alexandroff topology	\bullet	\circ	\bullet	\circ	\bullet	\bullet	[1]

Dikranjan *et al.* [6] show that the class of generalized closure space (X, cl) satisfying (K0), (K1), (K2), and (K4) forms a topological category. It is well known that the Čech closure spaces, which satisfy (K0), (K1), (K2), and (K3), are identical to the pretopological spaces which also form a topological category, see e.g. [15].

An extensive textbook by Eduard Čech [4] demonstrates that much of the classical theory of point set topology remains intact in pretopological spaces, i.e., when the assumption that the closure is idempotent is dropped. As a consequence, the notions of open and closed sets play little actual role in this theory — quite in contrast to the way this mathematical theory is usually taught.

Further generalizations have received much less attention. In neighborhood spaces, for example, much of the hierarchy of separation axioms is still intact [18], while other important results, such as Urysohn's lemma, fail [19]. As we shall see in the following section, one has to be even more careful in the case of isotonic spaces.

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