Analysis of the $(\mu/\mu_I, \lambda)$-CSA-ES with Repair by Projection Applied to a Conically Constrained Problem

Patrick Spettel
Research Center Process and Product Engineering, Vorarlberg University of Applied Sciences, Dornbirn, 6850, Austria

Hans-Georg Beyer
Department of Computer Science, Research Center Process and Product Engineering, Vorarlberg University of Applied Sciences, Dornbirn, 6850, Austria

SUPPLEMENTARY MATERIAL

Appendix
This appendix contains material supplementing the main text. Appendix A contains a detailed derivation for the $s_{\infty}$ mean value difference equation. Additional figures comparing the derived approximations with simulations for the iterative system and steady state considerations are presented in Appendix B. Appendix C provides plots of further cumulation parameter influence investigations. Finally, Appendix D shows plots that visualize the influences of the damping parameter.
A Derivation of a Mean Value Difference Equation for $s_\circ$

For $s_\circ$, a mean value difference equation can be derived using the update rule from Line 19 of Algorithm 1 and considering Equation (7). To begin with,

$$s_{\circ}^{(g+1)} = \frac{1}{r^{(g+1)}} \sum_{k=2}^{N} (x^{(g+1)})_k (s^{(g+1)})_k$$

(A.1)

$$= \frac{1}{r^{(g+1)}} \sum_{k=2}^{N} [x^{(g)}_k + \sigma^{(g)}(\bar{z}^{(g)})_k] \left[ (1-c)(s^{(g)})_k + \sqrt{\mu_c(2-c)}(\bar{z}^{(g)})_k \right]$$

(A.2)

$$= \frac{1}{r^{(g+1)}} \sum_{k=2}^{N} (1-c) \left[ (x^{(g)})_k (s^{(g)})_k + \sigma^{(g)}(\bar{z}^{(g)})_k (s^{(g)})_k \right]$$

$$+ \frac{1}{r^{(g+1)}} \sum_{k=2}^{N} \sqrt{\mu_c(2-c)} \left[ (x^{(g)})_k (\bar{z}^{(g)})_k + \sigma^{(g)}(\bar{z}^{(g)})_k (\bar{z}^{(g)})_k \right]$$

(A.3)

can be derived. Equation (A.3) can further be rewritten by the introduction of $z^{(g)}_{\circ} := \frac{1}{r^{(g)}} \sum_{k=2}^{N} (x^{(g)})_k (\bar{z}^{(g)})_k$ (similar to Equation (7)) and use of Equation (14) resulting in

$$s_{\circ}^{(g+1)} = \frac{r^{(g)}}{r^{(g+1)}} (1-c) \left[ s^{(g)}_{\circ} + \frac{\sigma^{(g)}}{N^*}(\bar{z}^{(g)})_2..N (s^{(g)})_2..N \right]$$

$$+ \frac{r^{(g)}}{r^{(g+1)}} \sqrt{\mu_c(2-c)} \left[ z^{(g)}_{\circ} + \frac{\sigma^{(g)}}{N^*} ||(\bar{z}^{(g)})_2..N||^2 \right].$$

(A.4)

For the fraction $r^{(g)}/r^{(g+1)}$, $r^{(g+1)}$ has to be derived. From the offspring generation and selection steps it follows that

$$r^{(g+1)} = \sqrt{\sum_{k=2}^{N} ((x^{(g)})_k + \sigma^{(g)}(\bar{z}^{(g)})_k)^2}$$

(A.5)

$$= \sqrt{\sum_{k=2}^{N} (x^{(g)})_k^2 + 2\sigma^{(g)}(x^{(g)})_k (\bar{z}^{(g)})_k + \sigma^{(g)}(\bar{z}^{(g)})_k^2}$$

(A.6)

$$= \sqrt{r^{(g)}^2 + 2\frac{\sigma^{(g)}}{N^*} r^{(g)} z^{(g)}_{\circ} + \frac{\sigma^{(g)}}{N^*} r^{(g)}^2 ||(\bar{z}^{(g)})_2..N||^2}$$

(A.7)

holds. Using the result from Equation (A.7),

$$\frac{r^{(g)}}{r^{(g+1)}} = \sqrt{\frac{1 + 2\frac{\sigma^{(g)}}{N^*} z^{(g)}_{\circ} + \frac{\sigma^{(g)}}{N^*} ||(\bar{z}^{(g)})_2..N||^2}{1 + 2\frac{\sigma^{(g)}}{N^*} z^{(g)}_{\circ} + \frac{\sigma^{(g)}}{N^*} ||(\bar{z}^{(g)})_2..N||^2}}$$

(A.8)

can be derived. For further simplification of Equation (A.8), asymptotic assumptions are made for $N \to \infty$. Because the mutation vector is corrected in case of projection (Line 11 in Algorithm 1), $(\bar{z}^{(g)})$ denotes the centroid of the $\mu$ best (w.r.t. fitness)
offspring mutation vectors after the projection step. Approximation of $\langle \tilde{z}^{(g)} \rangle$ for the asymptotic case by its value before projection and selection yields a normal distribution for $(\langle \tilde{z}^{(g)} \rangle)_k = \frac{1}{\mu} \sum_{m=1}^{\mu} (\tilde{z}_{m:k})_k \sim N(0, \frac{1}{\mu}) = \frac{1}{\sqrt{\mu}} N(0, 1)$, which follows by the properties of a sum of normal distributed random variables.

Hence, $||((\langle \tilde{z}^{(g)} \rangle)_{2..N})||^2$ can be approximated by a $\chi^2$ distribution with $N - 1$ degrees of freedom. As the expected value of the $\chi^2$ distribution corresponds to its number of degrees of freedom,

$$\frac{||((\langle \tilde{z}^{(g)} \rangle)_{2..N})||^2}{N} \sim \frac{1}{\mu}$$

(A.9)

follows for $N \to \infty$ by the law of large numbers. With Equation (A.9) and the assumptions $N \gg 2\sigma^{(g)} z_{\circ}^{(g)}$ and $\mu N \gg \sigma^{(g)}$, 

$$\frac{r^{(g)}}{r^{(g+1)}} \sim 1$$

(A.10)

follows. Making use of Equation (A.10) and Equation (A.9), Equation (A.4) can be simplified for the asymptotic case $N \to \infty$ yielding

$$s_{\circ}^{(g+1)} \simeq (1 - c)s_{\circ}^{(g)} + (1 - c) \frac{\sigma^{(g)} z_{\circ}^{(g)}}{N} (\langle \tilde{z}^{(g)} \rangle)_{2..N}^{T}(s^{(g)})_{2..N} + \sqrt{\mu c(2 - c)z_{\circ}^{(g)}} + \sqrt{\mu c(2 - c)} \frac{\sigma^{(g)}}{\mu}.$$  

(A.11)

Taking expected values of Equation (A.11) with $E[s_{\circ}^{(g+1)}] := s_{\circ}^{(g+1)}$ results in

$$\frac{E[s_{\circ}^{(g+1)}]}{s_{\circ}^{(g+1)}} \simeq (1 - c)s_{\circ}^{(g)} + (1 - c) \frac{\sigma^{(g)} z_{\circ}^{(g)}}{N} E[(\langle \tilde{z}^{(g)} \rangle)_{2..N}^{T}(s^{(g)})_{2..N} ] + \sqrt{\mu c(2 - c)E[z_{\circ}^{(g)}]} + \sqrt{\mu c(2 - c)} \frac{\sigma^{(g)}}{\mu}.$$  

(A.12)

To treat Equation (A.12) further, $E[(\langle \tilde{z}^{(g)} \rangle)_{2..N}^{T}(s^{(g)})_{2..N}]$ and $E[z_{\circ}^{(g)}]$ need to be derived. For $E[(\langle \tilde{z}^{(g)} \rangle)_{2..N}^{T}(s^{(g)})_{2..N}]$, $(\langle \tilde{z}^{(g)} \rangle)_{2..N}$ is decomposed into a vector in direction of the parental individual’s $2..N$ components $e_{\circ}^{(g)}$ and in a direction $e_{\circ}^{(g)}$ that is orthogonal to $e_{\circ}^{(g)}$, i.e., $e_{\circ}^{(g)} e_{\circ}^{(g)T} = 0$. Further, in the following the assumption is made that those direction vectors are unit vectors, i.e., $||e_{\circ}^{(g)}|| = 1$ and $||e_{\circ}^{(g)}|| = 1$. Therefore, $(\langle \tilde{z}^{(g)} \rangle)_{2..N}$ can be written as

$$((\tilde{z}^{(g)}))_{2..N} = z_{\circ}^{(g)} e_{\circ}^{(g)} + z_{\circ}^{(g)} e_{\circ}^{(g)},$$

(A.13)

where $z_{\circ}^{(g)}$ and $z_{\circ}^{(g)}$ are the projections of the mutation vector in direction of $e_{\circ}^{(g)}$ and $e_{\circ}^{(g)}$, respectively. Using Equation (A.13),

$$(\langle \tilde{z}^{(g)} \rangle)_{2..N}^{T}(s^{(g)})_{2..N} = z_{\circ}^{(g)} e_{\circ}^{(g)T} (s^{(g)})_{2..N} + z_{\circ}^{(g)} e_{\circ}^{(g)} e_{\circ}^{(g)T} (s^{(g)})_{2..N} + z_{\circ}^{(g)} e_{\circ}^{(g)} e_{\circ}^{(g)T} (s^{(g)})_{2..N}$$

(A.14)

follows. Note that $e_{\circ}^{(g)T} (s^{(g)})_{2..N}$ corresponds to the definition in Equation (7). Taking into account the statistical independence of the cumulated path vector and the mutation
in the current generation, taking expectation results in

\[
E[(\bar{z}^{(g)})_{2..N}(s^{(g)})_{2..N}] = E[z^{(g)}_{\infty}]E[s^{(g)}_{\infty}] + E[z^{(g)}_{\infty}]E[e^{(g)}_0^T(s^{(g)})_{2..N}] \\
= E[z^{(g)}_{\infty}]s^{(g)}_{\infty}.
\]  

(A.15)

(A.16)

Note that \(E[z^{(g)}_{\infty}]\) vanishes because the mutations in direction \(e^{(g)}_0\) are isotropic and selectively neutral. Hence, the second summand of Equation (A.15) is 0 in expectation.

\(E[z^{(g)}_{\infty}]\) can be calculated from the progress rate of the quadratic distance from the cone axis. It writes

\[
\varphi^{(g)}_{r^2} := E[\rho^{(g)^2} - \rho^{(g+1)^2}] = \rho^{(g)^2} - E[\rho^{(g+1)^2}] \\
\approx \rho^{(g)^2} - \{P_{\text{feas}}(x^{(g)}, r^{(g)}, \sigma^{(g)})E[\langle q_r \rangle^2_{\text{feas}}] + [1 - P_{\text{feas}}(x^{(g)}, r^{(g)}, \sigma^{(g)})]E[\langle q_r \rangle^2_{\text{infeas}}]\} \\
\]  

(A.17)

(A.18)

where \(\langle q_r \rangle\) denotes the distance from the cone boundary of the centroid after projection (cf. Line 22 of Algorithm 1). Expressions for \(E[\langle q_r \rangle^2_{\text{feas}}]\) and \(E[\langle q_r \rangle^2_{\text{infeas}}]\) have already been derived in Appendix D in the supplementary material of Spettel and Beyer (2018b). The used Taylor approximation in Equation (D.157) of that work allows using the square of Equation (D.165) for the feasible case yielding

\[
E[\langle q_r \rangle^2_{\text{feas}}] \approx \rho^{(g)^2} + \frac{\sigma^{(g)^2}}{\mu}(N - 1).
\]  

(A.19)

Similarly, the Taylor expansion used in Equation (D.172) of that work allows using the square of Equation (D.217) as an approximation for the infeasible case. It reads

\[
E[\langle q_r \rangle^2_{\text{infeas}}] \approx \frac{E[\langle q \rangle^2_{\text{infeas}}]}{\xi} \left(1 + \frac{\sigma^{(g)^2}}{\mu N/\xi} \right) \left(1 + \frac{\sigma^{(g)^2}}{N/\xi} \right).
\]  

(A.20)

Using Equation (A.19) and Equation (A.20),

\[
\varphi^{(g)}_{r^2} \approx \rho^{(g)^2} - \{P_{\text{feas}}(x^{(g)}, r^{(g)}, \sigma^{(g)}) \left[ \rho^{(g)^2} + \frac{\sigma^{(g)^2}}{\mu}(N - 1) \right] \\
+ [1 - P_{\text{feas}}(x^{(g)}, r^{(g)}, \sigma^{(g)})] \left[ \frac{E[\langle q \rangle^2_{\text{infeas}}]}{\xi} \left(1 + \frac{\sigma^{(g)^2}}{\mu N/\xi} \right) \left(1 + \frac{\sigma^{(g)^2}}{N/\xi} \right) \right]\} \\
\]  

(A.21)

follows, where a closed-form approximation

\[
E[\langle q \rangle^2_{\text{infeas}}] \approx \frac{\xi}{1 + \xi} \left(x^{(g)} + \bar{r}/\sqrt{\xi}\right) - \frac{\xi}{1 + \xi} \sqrt{\sigma^{(g)^2} + \sigma^{(g)^2}/\xi} \sqrt{\mu_{\lambda}}
\]  

(A.22)

has been derived in Spettel and Beyer (2018b) as well (refer to the derivations leading to Equation (C.149) in Appendix C in the supplementary of that work for the details). With Equation (A.21) and Equation (A.22), \(\varphi^{(g)}_{r^2}\) can be computed for a given state of the system. The goal is now to express \(\varphi^{(g)}_{r^2}\) in terms of \(E[z^{(g)}_{\infty}]\). Subsequently solving
for $E[z^{(g)}]$ allows then to compute its value. Using Equation (A.17) and Equation (A.7) with Equation (A.9), $\varphi_{x_2}^{(g)}$ can alternatively be written as

$$\varphi_{x_2}^{(g)} = \frac{r^{(g)}_2}{2\sigma^{(g)}r^{(g)}_2} - \frac{\sigma^{(g)}N_2}{2\sigma^{(g)}r^{(g)}_2} = \frac{N\varphi_{x_2}^{(g)}}{2\sigma^{(g)}r^{(g)}_2} - \frac{\sigma^{(g)}\mu}{2\mu}. \tag{A.25}$$

Reinsertion of Equation (A.16) and Equation (A.25) into Equation (A.12) yields

$$s^{(g+1)}_\circ \approx (1-c) \left( \frac{\sigma^{(g)}}{N} E[z^{(g)}] \right)^{g} + \frac{\sqrt{\mu(2-c)}(E[z^{(g)}] + \frac{\sigma^{(g)}}{\mu})}{s^{(g)}_\circ} \left( 1 + \frac{\sigma^{(g)}}{N} \frac{N\varphi_{x_2}^{(g)}}{2\sigma^{(g)}r^{(g)}_2} - \frac{\sigma^{(g)}\mu}{2\mu} \right) s^{(g)}_\circ \tag{A.26}$$

$$\approx (1-c) \left( 1 + \frac{\sigma^{(g)}}{N} \left( - \frac{N\varphi_{x_2}^{(g)}}{2\sigma^{(g)}r^{(g)}_2} + \frac{\sigma^{(g)}\mu}{2\mu} \right) s^{(g)}_\circ \right) + \frac{\sqrt{\mu(2-c)}(E[z^{(g)}] + \frac{\sigma^{(g)}}{\mu})}{s^{(g)}_\circ} \left( - \frac{N\varphi_{x_2}^{(g)}}{2\sigma^{(g)}r^{(g)}_2} + \frac{\sigma^{(g)}\mu}{2\mu} \right). \tag{A.27}$$

B Additional Results Comparing the Derived Approximations with Simulations

Figures 7 to 12 show the mean value dynamics of the $(3/3,1,10)$-CSA-ES applied to the conically constrained problem with different parameters as indicated in the title of the subplots. The plots are organized into three rows and two columns. The first two rows show the $x$ (first row, first column), $r$ (first row, second column), $\sigma$ (second row, first column), and $\sigma^*$ (second row, second column) dynamics. The third row shows $x$ and $r$ converted into each other by $\sqrt{\xi}$. The third row shows that after some initial phase, the ES transitions into a stationary state. In this steady state, the ES moves along the cone boundary. This becomes clear in the plots because the equation for the cone boundary is $r = x/\sqrt{\xi}$ or equivalently $x = r\sqrt{\xi}$. In the first two rows, the solid line has been generated by averaging 100 real runs of the ES. The dashed line has been determined by iterating the mean value iterative system of Section 3.1.6 with one-generation experiments for $\varphi_x^{(g)}$, $\varphi_x^{(g)*}$, $\varphi_r^{(g)}$, $\varphi_r^{(g)*}$, and $\varphi_{x_2}^{(g)}$. The dotted lines have been computed by iterating the mean value iterative system with the derived approximations as indicated in the derivations leading to the equations in Section 3.1.6 for $\varphi_x^{(g)}$, $\varphi_x^{(g)*}$, $\varphi_r^{(g)}$, $\varphi_r^{(g)*}$, and $\varphi_{x_2}^{(g)}$. Due to the approximations used it is possible that in a generation $g$, the iteration of the mean value iterative system yields infeasible $(x^{(g)}, r^{(g)})^T$. In such cases, the particular $(x^{(g)}, r^{(g)})^T$ have been projected back and projected values have been used in the further iterations.

Figure 13 and Figure 14 show plots of the steady state computations using the numerical solution of Equation (62) for $\sigma^{(g)}_x$, which has been compared to real ES runs. The results for $\varphi_x^*$ and $\varphi_r^*$ have been determined by using the numerically determined
Figure 7: Mean value dynamics closed-form approximation and real-run comparison of the (3/3r, 10)-CSA-ES with repair by projection applied to the conically constrained problem. (Part 1)
Figure 8: Mean value dynamics closed-form approximation and real-run comparison of the $(3/3, 10)$-CSA-ES with repair by projection applied to the conically constrained problem. (Part 2)
Figure 9: Mean value dynamics closed-form approximation and real-run comparison of the (3/3J, 10)-CSA-ES with repair by projection applied to the conically constrained problem. (Part 3)
Figure 10: Mean value dynamics closed-form approximation and real-run comparison of the $\langle 3/3 \rangle$, 10)-CSA-ES with repair by projection applied to the conically constrained problem. (Part 4)
Figure 11: Mean value dynamics closed-form approximation and real-run comparison of the (3/3, 10)-CSA-ES with repair by projection applied to the conically constrained problem. (Part 5)
Figure 12: Mean value dynamics closed-form approximation and real-run comparison of the (3/3), 10)-CSA-ES with repair by projection applied to the conically constrained problem. (Part 6)
steady state $\sigma_{*}^{ss}$ values with Equation (48). The approximations for $\left(\frac{\sqrt{\xi}}{\sqrt{r}}\right)_{ss}$ have been determined by evaluating Equation (47). The values for the points denoting the experiments have been determined by computing the averages of the particular values in real ES runs.

Figures 15 to 22 show plots of the steady state computations. Results computed by Equation (74) for Figures 15 to 18 and Equation (87) for Figures 19 to 22 have been compared to real $(\mu/\mu_I, \lambda)$-CSA-ES runs for $\lambda = 10$. Different sets of $\mu$ values are considered. Figures 15, 16, 19 and 20 show the results for $\mu \in \{1, 2, 3\}$ and Figures 17, 18, 21 and 22 show the results for $\mu \in \{4, 6, 8\}$. The values for the points denoting the approximations have been determined by computing the normalized steady state mutation strength $\sigma_{*}^{ss}$ using Equation (74) for Figures 15 to 18 and Equation (87) for Figures 19 to 22, for different values of $\xi$. The results for $\varphi_{*}^{ss}$ and $\varphi_{r}^{ss}$ have been determined by using the computed steady state $\sigma_{*}^{ss}$ values with Equation (48). The approximations for $\left(\frac{x}{\sqrt{\xi}}\right)_{ss}$ have been determined by evaluating Equation (47). The values for the points denoting the experiments have been determined by computing the averages of the particular values in real ES runs. The figures show that the derived expressions get better for larger values of $\xi$ and $N$. Again, the deviations for small $\xi$ are due to approximations in the derivation of the local progress rates. The deviations for large $N$ stem from the use of asymptotic assumptions $N \to \infty$.

C Additional Results of the Cumulation Parameter’s Influence

Figure 23 shows the influence of the cumulation parameter for the considered $(3/3I, 10)$-CSA-ES for the dimensions 1000 and 10000. They extend the results shown in the main text. The steady state normalized $x$ progress rate (Equation (48)) is plotted against the steady state normalized mutation strength for different values of $\xi$. Steady state values achieved for different settings of $c$ with $D = \frac{1}{N}$ have been computed numerically (using Equation (62) with Equations (48) and (53) to (55)) and are shown with the filled black markers.

D Additional Investigations of the Damping Parameter

Figures 24 to 27 show the influence of the damping parameter for the considered $(3/3I, 10)$-CSA-ES for the dimensions 40, 400, 1000, and 10000. The steady state normalized $x$ progress rate (Equation (48)) is plotted against the steady state normalized mutation strength for different values of $\xi$. Steady state values achieved for different settings of $D$ with $c = \frac{1}{\sqrt{N}}$ (Figures 24 and 25) and $c = \frac{1}{N}$ (Figures 26 and 27) have been computed numerically (using Equation (62) with Equations (48) and (53) to (55)) and are shown with the filled black markers.

The plots show that for $c = 1/\sqrt{N}$ the damping parameters considered only have an effect for dimension 40. In contrast to that, for $c = 1/N$, the damping parameters considered can increase the steady state $x$ progress for all the dimensions and values of $\xi$ shown.
Figure 13: Steady state comparison of the \((\mu/\mu_1, \lambda)\)-CSA-ES with repair by projection applied to the conically constrained problem for \(c = \frac{1}{N}, D = \frac{1}{\xi}\), \(\lambda = 10, \mu \in \{1, 2, 3\}\). The numerical solution of Equation (62) for \(\sigma^*_{ss}\) has been considered for these subplots. (Part 1)
The numerical solution of Equation (62) for $\sigma^{*}_{ss}$ has been considered for these subplots.

Figure 14: Steady state comparison of the $(\mu/\mu_{I}, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{N}, D = \frac{1}{2}, \lambda = 10, \mu \in \{1, 2, 3\}$. The numerical solution of Equation (62) for $\sigma^{*}_{ss}$ has been considered for these subplots. (Part 2)
Figure 15: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_1, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{\sqrt{N}}$, $D = \frac{1}{2}$, $\lambda = 10$, $\mu \in \{1, 2, 3\}$. (Part 1)
Figure 16: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_1, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{\sqrt{N}}, \lambda = 10, \mu \in \{1, 2, 3\}$. (Part 2)
Figure 17: Steady state closed-form approximation and real run comparison of the $(\mu/\mu, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{\sqrt{N}}, D = \frac{1}{2}, \lambda = 10, \mu \in \{4, 6, 8\}$. (Part 1)
Figure 18: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_1, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{\sqrt{N}}, D = \frac{1}{\tau}, \lambda = 10, \mu \in \{4, 6, 8\}$. (Part 2)
Figure 19: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_1, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{N}, D = \frac{1}{\xi}$, $\lambda = 10$, $\mu \in \{1, 2, 3\}$, (Part 1)
Figure 20: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_I, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{N}, D = \frac{1}{e}, \lambda = 10, \mu \in \{1, 2, 3\}$. (Part 2)
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Figure 21: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_t, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{N}, D = \frac{1}{\tilde{c}}, \lambda = 10, \mu \in \{4, 6, 8\}$. (Part 1)
Figure 22: Steady state closed-form approximation and real run comparison of the $(\mu/\mu_1, \lambda)$-CSA-ES with repair by projection applied to the conically constrained problem for $c = \frac{1}{\lambda}, D = \frac{1}{\xi}, \lambda = 10, \mu \in \{4, 6, 8\}$. (Part 2)
Figure 23: Influence of the cumulation parameter for the considered $(3/3_I, 10)$-CSA-ES for $N = 1000$ (top) and $N = 10000$ (bottom). For different values of $\xi$, the steady state progress rate is plotted against the steady state normalized mutation strength. The filled black markers indicate the steady state values achieved by different $c$ parameter settings with $D = 1/c$. 
Figure 24: Influence of the damping parameter for the considered \((3/3_I, 10)\)-CSA-ES for \(N = 40\) (top) and \(N = 400\) (bottom). For different values of \(\xi\), the steady state progress rate is plotted against the steady state normalized mutation strength. The filled black markers indicate the steady state values achieved by different \(D\) parameter settings with \(c = 1/\sqrt{N}\).
Figure 25: Influence of the damping parameter for the considered (3/3, 10)-CSA-ES for $N = 1000$ (top) and $N = 10000$ (bottom). For different values of $\xi$, the steady state progress rate is plotted against the steady state normalized mutation strength. The filled black markers indicate the steady state values achieved by different $D$ parameter settings with $c = 1/\sqrt{N}$.
Figure 26: Influence of the damping parameter for the considered $(3/3I, 10)$-CSA-ES for $N = 40$ (top) and $N = 400$ (bottom). For different values of $\xi$, the steady state progress rate is plotted against the steady state normalized mutation strength. The filled black markers indicate the steady state values achieved by different $D$ parameter settings with $c = 1/N$. 
Figure 27: Influence of the damping parameter for the considered \( (3/3/1, 10) \)-CSA-ES for \( N = 1000 \) (top) and \( N = 10000 \) (bottom). For different values of \( \xi \), the steady state progress rate is plotted against the steady state normalized mutation strength. The filled black markers indicate the steady state values achieved by different \( D \) parameter settings with \( c = 1/N \).