Notes to frictionless case of the model in

**Sequentiality versus Simultaneity:**

**Interrelated Factor demand**

April 19, 2013

Magne K. Asphjell

Wilko Letterie

Øivind A. Nilsen

Gerard A. Pfann

**Derivation serial correlation investment and hiring rate**

Equations:

1. \( A_t = A_t^{ρ} \alpha^ε \) implying \( a_t = \ln A_t = ρ \ln A_{t-1} + ε_t = ρu_{t-1} + e_t \)

NB \( E(e^ε) = e^{σ^2/2} \) with \( ε_t \sim N(0, σ^2) \)

2. \( K_{t+1} = (1 - δ^K) \cdot K_t + I_t \) implying that \( \frac{I_t}{K_t} = \frac{K_{t+1} - K_t}{K_t} + δ^K \)

3. \( L_{t+1} = (1 - δ^L) \cdot L_t + H_t \) implying that \( \frac{H_t}{L_t} = \frac{L_{t+1} - L_t}{L_t} + δ^L \)

These equations also mean that net and gross change rates have the same covariances / correlations if depreciation and autonomous quits are fixed.

With frictionless inputs and time to build the firm maximizes each period:

4. \( \max_{i_t, h_t} E[A_{i_t} \cdot \left( (1 - δ^K) \cdot K_t + I_t \right)^α \left( (1 - δ^L) \cdot L_t + H_t \right)^β - p^I I_t - p^H H_t] \)

This gives two first order conditions

5. \( αE(A_{i_t})K_{i_t}^{α-1}L_t^{β} - p^I = 0 \)

6. \( βE(A_{i_t})K_{i_t}^{α}L_t^{β-1} - p^H = 0 \)
Therefore,

\[ K_{t+1} = \left[ \frac{\alpha}{p^I} E(A_{t+1}) L_{t+1}^\beta \right]^{\frac{1}{1-\alpha}} \]

\[ L_{t+1} = \left[ \frac{\beta}{p^H} E(A_{t+1}) K_{t+1}^\alpha \right]^{\frac{1}{1-\beta}} \]

Hence,

\[
L_{t+1} = \left[ \frac{\beta}{p^H} E(A_{t+1}) \left( \left[ \frac{\alpha}{p^I} E(A_{t+1}) L_{t+1}^\beta \right]^{\frac{1}{1-\alpha}} \right)^\alpha \right]^{\frac{1}{1-\beta}} \\
= \frac{\beta}{p^H} \left( \frac{\alpha}{p^I} \right)^{\frac{\alpha}{1-\alpha(1-\beta)}} E(A_{t+1})^{1-\beta} \cdot L_{t+1}^{\frac{\alpha}{1-\alpha(1-\beta)}}
\]

\[ L_{t+1} \propto E(A_{t+1})^{\frac{1}{1-\alpha-\beta}} \]

Thus,

\[ L_{t+1} \propto E(A_{t+1})^{\frac{1}{1-\alpha-\beta}} \]

The constant terms are disregarded here as they play no role later.

Likewise for capital

\[ K_{t+1} \propto E(A_{t+1})^{\frac{1}{1-\alpha-\beta}} \]

For labour it is found

\[ \ln L_{t+1} = \frac{1}{1-\alpha-\beta} \ln E(A_{t+1}) + \text{const} = \frac{1}{1-\alpha-\beta} \ln E(A_t^{\rho} e^{\varepsilon_t}) + \text{const} = \frac{\rho}{1-\alpha-\beta} a_t + \text{const} \]

As a consequence,

\[ \ln L_{t+1} = \frac{\rho^2}{1-\alpha-\beta} \cdot a_{t-1} + \frac{\rho}{1-\alpha-\beta} \varepsilon_t + \text{const} = \frac{\rho^3}{1-\alpha-\beta} \cdot a_{t-2} + \frac{\rho^2}{1-\alpha-\beta} \varepsilon_{t-1} + \frac{\rho}{1-\alpha-\beta} \varepsilon_t + \text{const} \]

\[ = \rho \cdot \ln L_t + \frac{\rho}{1-\alpha-\beta} \varepsilon_t + \text{const} \]

To save a bit on notation this is rewritten as

\[ x_t = \rho x_{t-1} + \eta_{t-1} + \text{const} \]
This means that $x_t - x_{t-1}$ is a growth rate as the underlying variable is in natural logs. Thus

$$x_t = \sum_{i=0}^{\infty} \rho^i \eta_{t-i} + \frac{\text{const}}{1 - \rho}$$

Therefore $E(x_t - x_{t-1}) = 0$ and the variance of the growth rate is

$$E(x_t - x_{t-1})^2 = E\left(\sum_{i=0}^{\infty} \rho^i \eta_{t-i} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right)^2 = E\left(\eta_{t-1} + \rho \sum_{i=0}^{\infty} \rho^i \eta_{t-i} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right)^2$$

$$= E\left(\eta_{t-1} + (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right)^2 = \sigma^2 \left(1 + \frac{(1 - \rho)^2}{(1 - \rho^2)}\right) = \sigma^2 \left(1 + \frac{(1 - \rho)}{1 + \rho}\right) = \sigma^2 \left(\frac{2}{1 + \rho}\right)$$

The first order covariance is

$$E(x_t - x_{t-1})(x_{t-1} - x_{t-2}) = E\left(\sum_{i=0}^{\infty} \rho^i \eta_{t-i-1} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right)$$

$$= E\left(\eta_{t-1} + (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right)$$

$$= E\left((\rho - 1) \eta_{t-2} + \rho (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-3}\right)$$

$$= \sigma^2 \left((\rho - 1) + \rho \frac{(1 - \rho)^2}{1 - \rho^2}\right) = \sigma^2 \left((\rho - 1) + \rho \frac{(1 - \rho)}{1 + \rho}\right) = -\sigma^2 \left(\frac{1 - \rho}{1 + \rho}\right)$$

Using equations (17) and (18) the first order correlation is obtained.

$$\text{Corr}(x_t - x_{t-1}, x_{t-1} - x_{t-2}) = \frac{-\sigma^2 \left(\frac{1 - \rho}{1 + \rho}\right)}{\sigma^2 \left(\frac{2}{1 + \rho}\right)} = \frac{-1 - \rho}{2}$$

Thus with $-1 < \rho < 1$, the expression in equation (19) is negative. This means that the first order serial correlations for the investment rate, $I/K$, and the hiring rate $H/L$, are negative. The second order covariance is
\[
E(x_t - x_{t-1}) (x_{t-2} - x_{t-3}) = E\left(\sum_{i=0}^{\infty} \rho^i \eta_{t-i-1} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2}\right) \left(\sum_{i=0}^{\infty} \rho^i \eta_{t-i-3} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4}\right) = \\
= E\left(\rho^3 \sum_{i=0}^{\infty} \rho^i \eta_{t-i-3} - \rho \sum_{i=0}^{\infty} \rho^i \eta_{t-i-2} \right) \left(\sum_{i=0}^{\infty} \rho^i \eta_{t-i-3} - \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4}\right) \\
= E\left(\rho^3 \eta_{t-3} - \rho \eta_{t-2} + \rho^3 \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4} - \rho^2 \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4}\right) \left(\eta_{t-3} + (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4}\right) = \\
(20) \\
= E\left(\rho (\rho - 1) \eta_{t-3} + \rho^2 (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4} \right) \left(\eta_{t-3} + (\rho - 1) \sum_{i=0}^{\infty} \rho^i \eta_{t-i-4}\right) = \\
= \sigma^2 \left(\rho (\rho - 1) + \rho^2 \left(\frac{1 - \rho}{1 + \rho}\right)^2\right) = \sigma^2 \left(\rho (\rho - 1) + \rho^2 \left(\frac{1 - \rho}{1 + \rho}\right)\right) = -\sigma^2 \rho \left(\frac{1 - \rho}{1 + \rho}\right)
\]

Hence,

\[
(21) \text{Corr}(x_t - x_{t-1}) (x_{t-2} - x_{t-3}) = \frac{-\sigma^2 \rho \left(\frac{1 - \rho}{1 + \rho}\right)}{\sigma^2 \left(\frac{2}{1 + \rho}\right)} = -\frac{1 - \rho}{2} \rho
\]

Thus, the second order correlation has the opposite sign of \(\rho\). For labour and capital the derivations are identical, and hence their correlation moments are the same.

**Derivation serial correlation sales growth rate**

Using equations (11) and (12) sales is given by

\[
22) Y_{t+1} = A_{t+1} K_{t+1}^{\alpha} L_{t+1}^{\beta} \propto A_{t+1} E(A_{t+1})^{\alpha} E(A_{t+1})^{\beta} = A_{t+1} E(A_{t+1})^{\alpha + \beta} 
\]

First define \(\gamma = \frac{\alpha + \beta}{1 - \alpha - \beta}\)

Hence,

\[
223) \ln Y_{t+1} = \ln A_{t+1} + \gamma \ln E(A_{t+1}) = a_{t+1} + \gamma \ln E(A_{t+1} e^{\varepsilon_{t+1}}) = a_{t+1} + \gamma a_t + d + \rho \ln Y_t + \varepsilon_{t+1} + \gamma \varepsilon_t + \tilde{d}
\]

So \(\ln Y_t\) is an ARMA(1,1) process. The covariance structure of such a process is given by Granger and Newbold (1987, p27). If \(\lambda_p = \text{cov}(\ln Y_t, \ln Y_{t-p})\) then
(24) \[ \lambda_0 = \frac{(1 + 2\gamma \rho^2 + \gamma^2 \rho^2)}{1 - \rho^2} \sigma_e^2 \]

(25) \[ \lambda_i = \frac{(1 + \gamma \rho^2)(1 + \gamma)\rho}{1 - \rho^2} \sigma_e^2 \]

(26) \[ \lambda_\tau = \rho \lambda_{\tau-1} \text{ for } \tau \geq 2 \]

Given that \( \ln Y_t \) is stationary, for the sales growth rate it is found that

(27) \[ E(\ln Y_t - \ln Y_{t-1})^2 = 2E(\ln Y_t)^2 - 2E(\ln Y_t \ln Y_{t-1}) = 2(\lambda_0 - \lambda_i) \]

\[ E(\ln Y_t - \ln Y_{t-1})(\ln Y_{t-1} - \ln Y_{t-2}) = 2E(\ln Y_t \ln Y_{t-1}) - E(\ln Y_t \ln Y_{t-2}) \]

\[ = 2\lambda_1 - \lambda_0 - \lambda_2 = (2 - \rho)\lambda_1 - \lambda_0 \]

(28) \[ E(\ln Y_t - \ln Y_{t-1})(\ln Y_{t-2} - \ln Y_{t-3}) = 2E(\ln Y_t \ln Y_{t-1}) - E(\ln Y_t \ln Y_{t-2}) - E(\ln Y_t \ln Y_{t-3}) \]

\[ = 2\lambda_2 - \lambda_1 - \lambda_3 = (2 - 1 - \rho^2)\lambda_1 \]

As a result,

(30) \[ Corr(\ln Y_t - \ln Y_{t-1})(\ln Y_{t-1} - \ln Y_{t-2}) = \frac{(2 - \rho)\lambda_1 - \lambda_0}{2(\lambda_0 - \lambda_i)} = \frac{(2 - \rho)(1 + \gamma \rho^2)(1 + \gamma)\rho - (1 + 2\gamma \rho^2 + \gamma^2 \rho^2)}{2(1 + 2\gamma \rho^2 + \gamma^2 \rho^2) - 2(1 + \gamma \rho^2)(1 + \gamma)\rho} \]

(31) \[ Corr(\ln Y_t - \ln Y_{t-1})(\ln Y_{t-2} - \ln Y_{t-3}) = \frac{(2\rho - 1 - \rho^2)\lambda_1}{2(\lambda_0 - \lambda_i)} = \frac{(2\rho - 1 - \rho^2)(1 + \gamma \rho^2)(1 + \gamma)\rho}{2(1 + 2\gamma \rho^2 + \gamma^2 \rho^2) - 2(1 + \gamma \rho^2)(1 + \gamma)\rho} \]

Hence, these serial correlations are a function of the parameters of the sales function \( \alpha \) and \( \beta \) through \( \gamma \) and the parameter \( \rho \).

---

1 The derivations for correlation structure of the sales growth rate are robust to a nonzero mean in \( \ln Y_t \), as employing the growth rate corrects for this.